An exact solution of a multi-state generalisation of the six-vertex model

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Abstract. An exact solution of a $2 \times n$ generalisation (two states for horizontal arrows and m states for vertical arrows with total arrow number conservation) of the six-vertex model is given using the commuting transfer matrix method. In the symmetric case, the free energy is a sum of $n-1$ terms each of which is the free energy of a six-vertex model such that the phase transition into the frozen ferroelectric state of the $2 \times n$ model is identical to that in the zero-field six-vertex model. For the six-vertex model we extend the commuting transfer matrix solution to the case of arbitrary vertex weights and provide a derivation of the Baxter parametrisation.

1. Introduction

Statistical mechanical models with bonds having more than two states have been studied extensively in the last few years (Stroganov 1979, Schultz 1981, 1983). In particular, Onody and Karowsky (1983) have considered a (arrow–reflection) symmetric ten-vertex model which they solve exactly using the commuting transfer matrix method (Baxter 1982). In their model the horizontal bonds have two states while the vertical bonds have three states. We present an explicit solution of a generalisation of the model with an arbitrary number of states for vertical bonds (and two for horizontal bonds) and have included horizontal and vertical electric fields. We find that the phase transitions in this model in the absence of fields are the same as those in the six-vertex model.

In the rest of this section we define the model that is studied in this paper and introduce the commuting transfer matrix method. In § 2 the commuting transfer matrix (CTM) method for the six-vertex model is generalised slightly to include the vertical field directly. We also present a derivation of the Baxter parametrisation for the vertex weights. In § 3 we solve the symmetric $2 \times n$ state model and obtain its free energy. Using the method of § 2 we include horizontal and vertical fields. Note that this is *not* the most general model for $n > 2$. The free energy has not been derived explicitly in this case. A few concluding remarks can be found in the last section.

Consider a square lattice of N rows and M columns on a torus. The configurations of the system are specified by the states of the bonds: the horizontal bonds have two states represented by an arrow pointing to the left or to the right; the vertical bonds have n states represented by $n - 2j + 1$ arrows with $j = 1, 2, \ldots, n$ (if $n - 2j + 1$ is positive the arrows point upward, if negative the arrows point downward). A vertex corresponds to the specification of the four bonds that meet at a lattice site. The allowed vertices

are defined by the 'ice condition': the number of incoming arrows equals the number of outgoing arrows. Of a total of $4n^2$ vertices this restriction allows $4n - 2$ vertices. For $n = 2$ this reduces to the six-vertex model (Lieb and Wu 1972). With each vertex is associated a statistical weight $\omega_r = \exp(-\beta\varepsilon_r)$ where ε_r is the energy of the rth vertex configuration and $\beta = 1/k_B T$.

The partition function Z is defined by

$$Z = \sum_C \prod_{i=1}^{M} \prod_{j=1}^{N} \omega_{V_{(i,j)}} \tag{1.1}$$

where $V_{(i,j)}$ is the vertex configuration at site (i, j) and the sum is over all configurations C of the lattice. As usual, Z can be written as

$$Z = \text{Tr } T^N \tag{1.2}$$

where T is the row-to-row transfer matrix. Following Baxter we write (see e.g., Jayaprakash and Sinha 1982 (to be referred to as JS) for details)

$$T = \text{Tr}_\mu = A + D \tag{1.3}$$

and

$$\mathscr{T} = \mathscr{L}_1 \mathscr{L}_2 \ldots \mathscr{L}_n = \begin{pmatrix} A & B \\ C & D \end{pmatrix}. \tag{1.4}$$

The fact that in μ space $\mathscr{T}$ is a 2×2 matrix is a consequence of allowing only two states for the horizontal bond which makes this problem a straightforward generalisation of the six-vertex case. The $\mathscr{L}_j$ are defined by

$$\mathscr{L}_j = \begin{pmatrix} \hat{\alpha}_j & \hat{\beta}_j \\ \hat{\gamma}_j & \hat{\delta}_j \end{pmatrix} \tag{1.5a}$$

where

$$\hat{\alpha}_j = I \otimes I \otimes \ldots \otimes I \otimes \hat{\alpha} \otimes I \otimes \ldots \otimes I \tag{1.5b}$$

etc, with $\hat{\alpha}$ in the jth place, I is an $n \times n$ identity matrix and $\hat{\alpha}$, $\hat{\beta}$, $\hat{\gamma}$ and $\hat{\delta}$ are $n \times n$ matrices which give the weights to go from the n states of the vertical arrow in the kth row to those of the $(k+1)$st row at a vertex. The specific model we solve is defined in figure 1 and in § 3. We also define, $\mathscr{L}$, for notational convenience, by

$$\mathscr{L} = \begin{pmatrix} \hat{\alpha} & \hat{\beta} \\ \hat{\gamma} & \hat{\delta} \end{pmatrix}. \tag{1.5c}$$

The free energy per site is given by

$$-\beta f = \lim_{M \to \infty} \frac{1}{M} \ln \Lambda \tag{1.6}$$

where Λ is the largest eigenvalue of T. This is calculated using the commuting transfer matrix method of Baxter. This remarkable method involves finding a suitable reparametrisation of the vertex weights $\omega_r(v)$ such that

$$[T(v), T(v')] = 0. \tag{1.7}$$

A sufficient condition for this is the existence of a non-trivial, invertible, numerical matrix $\mathscr{R}$ such that

$$\mathscr{R}(v, v')\mathscr{L}(v) \otimes \mathscr{L}(v') = \mathscr{L}(v') \otimes \mathscr{L}(v)\mathscr{R}(v, v') \tag{1.8}$$

where $\mathcal{L}(v)$ and $\mathcal{L}(v')$ are 2×2 matrices in μ space and μ' space respectively and $\mathcal{R}$ is a 4×4 matrix in the direct product space. Equation (1.8) implies

$$\mathcal{R}(v, v')\mathcal{T}(v) \otimes \mathcal{T}(v') = \mathcal{T}(v') \otimes \mathcal{T}(v)\mathcal{R}(v, v'). \tag{1.9}$$

For a specific choice of $\mathcal{R}$ that obeys (1.8), (1.9) leads to commutation relations between the operators A, B, C and D in (1.4). These will be used to determine the eigenvalues and eigenvectors of $T = A + D$.

The most general form of the matrix $\mathcal{R}$ needed in this paper is given by

$$\mathcal{R} = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & e & 0 \\ 0 & f & c & 0 \\ 0 & 0 & 0 & d \end{pmatrix}. \tag{1.10}$$

Note that $\mathcal{R}$ is determined up to a scale factor which is fixed by an appropriate normalisation, usually $a = 1$. Then (1.9) gives the 'commutation relations' between $A(v) \ldots D(v)$ and $A(v') \ldots D(v')$. Here we record only the ones used for the construction of eigenvectors and eigenvalues of $T(v)$. These are

$$[A(v), A(v')] = [B(v), B(v')] = [C(v), C(v')] = [D(v), D(v')] = 0$$

$$A(u)B(v) = \frac{1}{f(v, u)} B(v)A(u) - \frac{b(v, u)}{f(v, u)} B(u)A(v) \tag{1.11}$$

$$D(u)B(v) = \frac{1}{f(u, v)} B(v)D(u) - \frac{c(u, v)}{f(u, v)} B(u)D(v).$$

In writing (1.11) we have assumed $a = d = 1$. If $a \neq d$, then $[B(v), B(u)] \neq 0$, and the method used here (see JS) for the construction of the eigenvectors of $T(v)$ fails. We also need the relation

$$\frac{b(u, v)}{f(u, v)} = -\frac{c(v, u)}{f(v, u)} \tag{1.12}$$

which in fact follows from

$$\mathcal{R}(u, v) = \mathcal{R}^{-1}(v, u). \tag{1.13}$$

Interchanging, v and v' in (1.8) leads to (1.13). All of these relations can be verified using the explicit form of $\mathcal{R}$ given in the later sections.

The eigenvectors of T are constructed from a suitable 'vacuum state' $|0\rangle$ using the 'creation operators' $B(v)$. Application of $B(v_1) \ldots B(v_n)$ on $|0\rangle$ leads to both 'wanted' and 'unwanted' terms. Setting these 'unwanted' terms equal to zero gives nonlinear equations involving v_i. These equations are identical to those obtained from the periodic boundary conditions in the original form of the Bethe ansatz method, and the solution of these equations gives the allowed values of v_i.

2. The unrestricted six-vertex model

The six-vertex model in the absence of the vertical field has been solved by the CTM method (see JS for details). This, however, is not a restriction since the free energy in the presence of the vertical field can be obtained by performing a Legendre

transformation on the free energy in the absence of the field. In this section we include the vertical field directly in the CTM solution. The same method works for the $2 \times n$ model discussed in § 3. (This also yields the most general commuting transfer matrices for the six-vertex model). As a useful aside we provide a derivation of the Baxter parametrisation of the vertex weights. More precisely, we show the need for the parametrisation in the solution of an integral equation required to calculate the free energy in the thermodynamic limit.

Using standard notation (JS and references cited therein), $\mathscr{L}$ defined in equation (1.5c) is given by†

$$\mathscr{L} = \begin{pmatrix} \omega_1 & 0 & 0 & 0 \\ 0 & \omega_4 & \omega_6 & 0 \\ 0 & \omega_5 & \omega_3 & 0 \\ 0 & 0 & 0 & \omega_2 \end{pmatrix}. \tag{2.1}$$

We can choose $\omega_5 = \omega_6$ without any loss of generality since we have imposed periodic boundary conditions. However, in models with staggered fields this condition cannot be satisfied and hence we leave the discussion general. $\mathscr{R}$ is chosen to have the form given in (1.10). Observe that in contrast to JS we have allowed for $e \neq f$. This general form has also been considered by Sogo *et al* (1982). As we show below this permits the inclusion of vertical fields.

Writing out the 16 matrix equations implied by (1.8) leads to various conditions. We use these to solve for ratios among $a, \ldots, f$, in terms of the ω_i and ω_i'. We exhibit below these relations for completeness:

$$\frac{a}{e} = \frac{\omega_1}{\omega_4} - \frac{\omega_5\omega_6}{X_2} \frac{\omega_4'}{\omega_4} \tag{2.2a}$$

$$\frac{a}{f} = \frac{\omega_4'}{\omega_1'} - \frac{\omega_5'\omega_6'}{X_1} \frac{\omega_1}{\omega_1'} \tag{2.2b}$$

$$\frac{d}{e} = \frac{\omega_3'}{\omega_2'} - \frac{\omega_5'\omega_6'}{X_2} \frac{\omega_2}{\omega_2'} \tag{2.2c}$$

$$\frac{d}{f} = \frac{\omega_2}{\omega_3} - \frac{\omega_5\omega_6}{X_1} \frac{\omega_3'}{\omega_3} \tag{2.2d}$$

$$\frac{e}{f} = \frac{X_2}{X_1} \tag{2.2e}$$

$$\frac{c}{e} = -\frac{\omega_5\omega_6'}{X_2} \tag{2.2f}$$

$$\frac{b}{e} = -\frac{\omega_6\omega_5'}{X_2} \tag{2.2g}$$

where

$$X_1 = \omega_1\omega_3' - \omega_3\omega_1'$$
$$X_2 = \omega_2\omega_4' - \omega_4\omega_2'. \tag{2.2h}$$

† Here we follow the usual conventions for the six-vertex model. This differs from the convention used in § 3 for the general $2 \times n$ state model, where the vertices ω_3 and ω_4 would be interchanged.

The consistency of this set of equations leads to :

Theorem. Two transfer matrices $T(\boldsymbol{\omega})$ and $T(\boldsymbol{\omega}')$ with vertex weights ω_i and ω_i commute, i.e.,

$$[T(\boldsymbol{\omega}), T(\boldsymbol{\omega}')] = 0 \tag{2.3a}$$

if the vertex weights satisfy the following conditions:

$$\frac{\omega_1\omega_2 + \omega_3\omega_4 - \omega_5\omega_6}{2(\omega_1\omega_2\omega_3\omega_4)^{1/2}} \equiv \Delta = \text{constant} \tag{2.3b}$$

$$\left(\frac{\omega_1\omega_4}{\omega_2\omega_3}\right) = \text{constant.} \tag{2.3c}$$

It is customary to 'parametrise' the vertex weights ω_i by a variable v, i.e., $\boldsymbol{\omega} = \boldsymbol{\omega}(v)$ and $\boldsymbol{\omega}' = \boldsymbol{\omega}(v')$, such that the transfer matrices commute for arbitrary values of v. The condition for this is that the left-hand sides of (2.3b) and (2.3c) be independent of v. A general solution which achieves this (for $\omega_5 = \omega_6$) is given by

$$\omega_1 = p_1\psi(v) \qquad \omega_2 = p_2\psi(v)$$

$$\omega_3 = p_3\phi(v) \qquad \omega_4 = p_4\phi(v) \tag{2.4}$$

$$\omega_5^2 = \omega_6^2 = \rho^2(\psi^2 + \phi^2 - 2\Delta\psi\phi) \equiv \rho^2 h^2(v)$$

where $p_1 p_2 = p_3 p_4 = \rho^2$ and $\psi(v)$ and $\phi(v)$ are arbitrary differentiable functions of v. Thus the vertex weights can be expressed in terms of the variable v and constants, Δ, ρ, E_x and E_y. The last two are defined by[†]

$$\exp(2\beta E_x) = \frac{\omega_1\omega_4}{\omega_2\omega_3} = \frac{p_1 p_4}{p_2 p_3} \tag{2.5a}$$

and

$$\exp(2\beta E_y) = \frac{\omega_1\omega_3}{\omega_2\omega_4} = \frac{p_1 p_3}{p_2 p_4}. \tag{2.5b}$$

One has the freedom to change the overall normalisation of the weights and thus only the ratio $\chi(v) = \psi(v)/\phi(v)$ is independent. Thus the $\mathcal{R}$ matrix can be determined up to an overall multiplicative factor using (2.2) and (2.4):

$$a = d \tag{2.6a}$$

$$b = c \tag{2.6b}$$

$$e/f = p_2 p_4 / p_1 p_3 \tag{2.5c}$$

$$\frac{a(v, v')}{e(v, v')} = \frac{\rho^2}{p_2 p_4} \frac{\psi\psi' + \phi\phi' - 2\Delta\psi\phi'}{\phi\psi' - \psi\phi'} \tag{2.6d}$$

$$\frac{b(v, v')}{e(v, v')} = \frac{\rho^2}{p_2 p_4} \frac{hh'}{\phi\psi' - \psi\phi'} \tag{2.6e}$$

where $\phi = \phi(v)$, $\phi' = \phi(v')$, etc.

[†] These equations differ by a factor of 2 from (2.11) in JS since we have taken each arrow to represent half a unit of polarisation in this paper.

The commutation relation (2.3*a*) can be written using the notation $T(\boldsymbol{\omega}) = T(\rho, \Delta, E_x, E_y, v) = T(v)$ as $[T(v), T(v')] = 0$. Then as has been described in detail in JS, the eigenstates of T can be constructed and the largest eigenvalue determined. The vacuum $|0\rangle$ is taken to be $e_1 \otimes e_1 \otimes \ldots \otimes e_1$ where $e_1 = \binom{1}{0}$ and the eigenvectors are

$$|v_1, v_2, \ldots, v_n\rangle = B(v_1)B(v_2) \ldots B(v_n)|0\rangle \tag{2.7a}$$

i.e.,

$$T|v_1, v_2, \ldots, v_n\rangle = \Lambda^{(n)}(v; v_1 \ldots v_n)|v_1, v_2, \ldots, v_n\rangle \tag{2.7b}$$

where

$$\Lambda^{(n)}(v; v_1 \ldots v_n) = (p_1\psi(v))^M \prod_{l=1}^{n} \frac{1}{f(v_l, v)} + (p_3\phi(v))^M \prod_{l=1}^{n} \frac{1}{f(v, v_l)} \tag{2.8}$$

and the v_j are determined by

$$\left(\frac{p_1\psi(v_j)}{p_3\phi(v_j)}\right)^M = \prod_{\substack{l=1 \\ l \neq j}}^{n} \frac{f(v_l, v_j)}{f(v_j, v_l)}. \tag{2.9}$$

In the limit $M \to \infty$ and $N \to \infty$ keeping n/M fixed, the v_j become dense and form a continuous distribution. This distribution $\rho(v)$ satisfies an integral equation (obtained by taking the logarithm of (2.9) and taking the thermodynamic limit) with a kernel given by (for the case $E_x = E_y = 0$)

$$K(v, v') = \frac{\partial}{\partial v}\theta(v, v') \tag{2.10a}$$

where

$$\exp(-i\theta(v, v')) = -\left(\frac{1 + \chi(v)\chi(v') - 2\Delta\chi(v)}{1 + \chi(v)\chi(v') - 2\Delta\chi(v')}\right). \tag{2.10b}$$

(Recall that $\chi(v) = \psi(v)/\phi(v)$). The Baxter parametrisation is obtained by requiring that $K(v, v')$ be a *difference* kernel, i.e., a function of $v - v'$ only:

$$i\frac{\partial}{\partial v}\ln\left(\frac{1 + \chi(v)\chi(v') - 2\Delta\chi(v)}{1 + \chi(v)\chi(v') - 2\Delta\chi(v')}\right) = K(v - v'). \tag{2.11}$$

For $v = v'$, this yields

$$\frac{\partial\chi}{\partial v} = +\frac{iK(0)}{2\Delta}(1 + \chi^2 - 2\Delta\chi(v)). \tag{2.12}$$

Letting $\Delta = \cos 2\eta$ (η may be complex) this equation is easily integrated to yield

$$\ln\frac{\chi - \exp(2i\eta)}{\chi - \exp(-2i\eta)} = -K(0)(\tan 2\eta)(v - v_0) \tag{2.13}$$

which is unique up to a rescaling ($K(0)$) and the choice of origin (v_0) for v. The form used by Lieb and Wu (1972)

$$\chi = \frac{\exp(\alpha) - \exp(i\mu)}{\exp(\alpha + i\mu) - 1} \tag{2.14}$$

can be obtained from (2.13) if we trade parameters (α, μ) for (v, η) by defining the right-hand side of (2.13) to be $\alpha + i\mu$.

Taking $\mu = 2\eta$ and $\alpha = -2iv$ leads to

$$\chi(v) = \sin(v + \eta)/\sin(v - \eta) \tag{2.15}$$

which is in fact Baxter's parametrisation†.

3. The $2 \times n$ state model

We consider the model with two states for horizontal bonds and n states for the vertical bonds represented by $n - 2j + 1$ arrows with $j = 1, \ldots, n$. The vertices allowed by the 'ice condition' and their weights are shown in figure 1. With appropriate labelling of
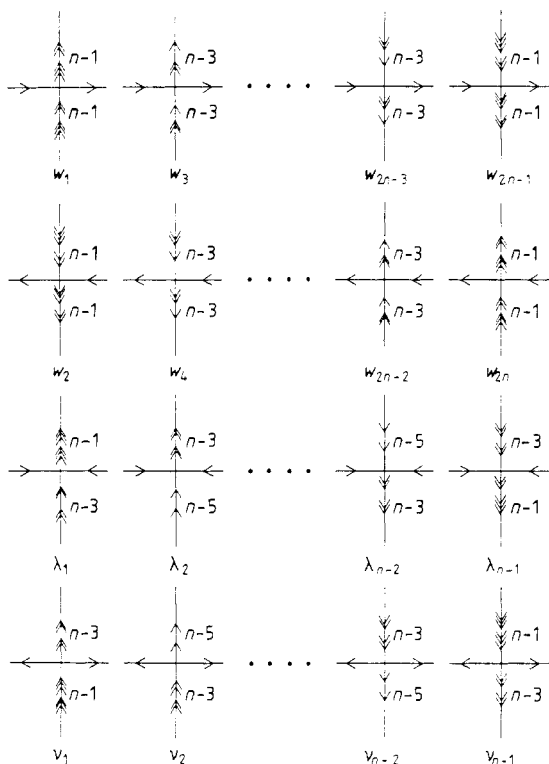

Figure 1. The vertices of the $2 \times n$ state model and their weights are shown here. In the odd-indexed w vertices both horizontal arrows point to the right; by the ice condition the number of vertical arrows going in must equal the number going out. The vertices are then ordered by beginning with the maximum possible number, $n - 1$, of arrows going upward and reducing the number by two until all $n - 1$ arrows point down. The corresponding even indexed vertices are obtained by reversing all the arrows as shown. In the λ vertices the horizontal arrows both point in; by the ice condition two more vertical arrows go out than come in. They are ordered in a similar fashion to the w vertices. The ν_j vertex is obtained by reversing all the arrows of the λ_{n-j} vertex.

† One can obtain our other parametrisation easily from (2.4) replacing the last equation by $\omega_5\omega_6 = \rho^2 h^2(v)$. Thus, $\omega_5\omega_6 = 1$ can be imposed by normalising (2.4). In the case $\omega_5 \neq \omega_6$, we have $b \neq c$, but the construction of eigenvectors etc works with trivial modifications.

the indices as shown in figure 1 the matrices $\hat{\alpha}$, $\hat{\beta}$, $\hat{\gamma}$ and $\hat{\delta}$ in (1.5c) are given by

$$\hat{\alpha} = \begin{pmatrix} w_1 & & & & 0 \\ & w_3 & & & \\ & & \ddots & & \\ 0 & & & & \\ & & & w_{2n-1} \end{pmatrix} \tag{3.1a}$$

$$\hat{\beta} = \begin{pmatrix} 0 & & & & 0 \\ \lambda_1 & 0 & & & \\ & \lambda_2 & 0 & & \\ & & 0 & \ddots & \\ & & & \lambda_{n-1} & 0 \end{pmatrix} \tag{3.1b}$$

$$\hat{\gamma} = \begin{pmatrix} 0 & \nu_1 & & & 0 \\ & 0 & \nu_2 & & \\ & & 0 & \ddots & \\ & & & 0 & \nu_{n-1} \\ & & & & 0 \end{pmatrix} \tag{3.1c}$$

$$\hat{\delta} = \begin{pmatrix} w_{2n} & & & 0 \\ & w_{2n-2} & & \\ & & \ddots & \\ 0 & & & w_2 \end{pmatrix}. \tag{3.1d}$$

For simplicity, we begin with the symmetric model (invariant under reversal of all arrows). In this case we have $w_{2i-1} = w_{2i} = \omega_i$ and $\lambda_j = \nu_{n-j}$. In addition, we impose the condition $\lambda_j = \nu_j$.

Since we are considering the zero-field model we choose $a = d$, $b = c$, and $e = f$ in the $\mathcal{R}$ matrix in (1.10). Then (1.8) leads to the following requirements (note that the choice $\hat{\beta}(v) = \hat{\beta}(v')$ is necessary and is used in the following):

$$a\omega_i' = b\omega_i + e\omega_{i+1}' \tag{3.2a}$$

$$a\omega_{i+1} = b\omega_{i+1}' + e\omega_i \tag{3.2b}$$

for $i = 1, 2, \ldots, n-1$ and

$$\begin{aligned} b(\omega_1'\omega_n - \omega_1\omega_n') &= e\lambda_1\nu_1 \\ b(\omega_j'\omega_{n-j+1} - \omega_j\omega_{n-j+1}') &= e(\lambda_j\nu_j - \lambda_{j-1}\nu_{j-1}) \\ b(\omega_n'\omega_1 - \omega_n\omega_1') &= -e\lambda_{n-1}\nu_{n-1}. \end{aligned} \tag{3.3}$$

'We have not analysed these equations in complete generality as in the case of (2.2). For $n = 3$, consistency requires

$$\frac{\omega_1 + \omega_3}{\omega_2} = \text{constant} \equiv 2 \cos 2\eta \tag{3.4a}$$

$$(\omega_1\omega_3 - \omega_2^2)\left(\frac{\omega_1 + \omega_3}{\omega_2}\right) = -\lambda_1\nu_1 \tag{3.4b}$$

(the parametrisation of Onody and Karowski (1983) satisfies these although they do

not give (3.4*a*) explicitly). The equations (3.2) imply

$$(\omega_i + \omega_{i+2})/\omega_{i+1} = \text{constant} \tag{3.5}$$

for $i = 1, 2, \ldots, n-2$. Guided by the known solutions for $n = 2$ and 3, we find that the following parametrisation satisfies (3.2) and (3.3):

$$\omega_j = \rho \sin[v + (n - 2j + 1)\eta] \qquad j = 1, \ldots, n. \tag{3.6}$$

The λ and ν are parametrised for $n = 2m + 1$ as

$$\lambda_i^2 = \nu_i^2 = \rho^2 \sin 2\eta \sum_{j=0}^{i-1} \sin 4(m-j)\eta \qquad i = 1, \ldots, m \tag{3.7a}$$

and for $n = 2m$ as

$$\lambda_i^2 = \nu_i^2 = \rho^2 \sin 2\eta \sum_{j=0}^{i-1} \sin(4m - 4j - 2)\eta \qquad i = 1, \ldots, m \tag{3.7b}$$

(the others are given by $\lambda_i = \lambda_{n-i}$).

For the $2 \times n$ state symmetric model, $T(v)$ and $T(v')$ commute and the elements of the matrix $\mathcal{R}$ are given by

$$a/e = \sin(2\eta + v - v')/\sin(v - v') \tag{3.8a}$$

$$b/e = \sin 2\eta/\sin(v - v'). \tag{3.8b}$$

Note that the parametrisation in (3.6)-(3.7) reduces to that of Baxter for $n = 2$, and essentially to that of Onody and Karowsky for $n = 3$. More significantly, for all n-state models, the $\mathcal{R}$ matrix is identical.

The inclusion of horizontal and vertical electric fields is straightforward. Unlike the six-vertex case, this is not the most general $(4n - 2)$ state model that can be considered. The horizontal field E_x does not affect λ_i and ν_i; the elements of $\hat{\alpha}$ and $\hat{\delta}$ are altered as follows:

$$\hat{\alpha}_{ij} \rightarrow \exp(\beta E_x)\hat{\alpha}_{ij} \tag{3.9a}$$

$$\hat{\delta}_{ij} \rightarrow \exp(-\beta E_x)\hat{\delta}_{ij}. \tag{3.9b}$$

Thus we have the parametrisation:

$$w_{2i-1} = p \sin[v + (n - 2i + 1)\eta] \tag{3.10a}$$

$$w_{2i} = q \sin[v + (n - 2i + 1)\eta] \qquad i = 1, \ldots, n \tag{3.10b}$$

where

$$p/q = \exp(2\beta E_x) \tag{3.11}$$

and $\{\lambda_i\}$, $\{\nu_i\}$ are given by (3.7*a*) or (3.7*b*) with $\rho^2 = pq$. In fact we can take

$$p = \rho \exp(\beta E_x) \qquad \text{and} \qquad q = \rho \exp(-\beta E_x) \tag{3.12}$$

and $\mathcal{R}$ remains unchanged.

The inclusion of a vertical field E_y is physically simple. Using periodic boundary conditions in the horizontal direction it is easy to show that the vertical polarisation P_v, ($\frac{1}{2}$ (number of up arrows − number of down arrows)), is conserved by the row-to-row transfer matrix and hence E_y contributes additively to the free energy. However, it is easy to include E_y directly within the commuting transfer matrix method by choosing

$e \neq f$ in $\mathcal{R}$. It can be verified that, by replacing e by $[\exp(-\beta E_y)]e$ and f by $[\exp(\beta E_y)]f$ in the zero-field result all the consistency conditions are satisfied for appropriate parametrisation of the vertex weights. This leads to the parametrisation including the fields:

$$w_{2i-1} = \rho \exp\{\beta[E_x + \tfrac{1}{2}(n - 2i + 1)E_y]\} \sin[v + (n - 2i + 1)\eta] \qquad (3.13a)$$

$$w_{2i} = \rho \exp\{-\beta[E_x + \tfrac{1}{2}(n - 2i + 1)E_y]\} \sin[v + (n - 2i + 1)\eta] \qquad (3.13b)$$

for $i = 1, 2, \ldots, n$, and

$$\lambda_i^2 = \nu_i^2 = \exp[\tfrac{1}{2}\beta(n - 2i)E_y](\lambda_i^2)_0 \qquad (3.13c)$$

where $(\lambda_i^2)_0$ is the zero-field expression given by $(3.7a)$ and $(3.7b)$. The corresponding $\mathcal{R}$ matrix is given by:

$$a = d = 1 \qquad (3.14a)$$

$$b(v, v') = c(v, v') = \frac{\sin 2\eta}{\sin(2\eta + v - v')} \qquad (3.14b)$$

$$e(v, v') \exp(\beta E_y) = f(v, v') \exp(-\beta E_y) = \frac{\sin(v - v')}{\sin(2\eta + v - v')}. \qquad (3.14c)$$

The construction of eigenvalues and eigenvectors of T proceeds as usual. The vacuum state oan be taken to be

$$|0\rangle = e_1 \otimes e_1 \otimes \ldots \otimes e_1, \qquad (3.15)$$

where

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

is a $n \times 1$ column vector. The eigenvectors are then given by

$$|v_1, v_2, \ldots, v_m\rangle = B(v_1) \ldots B(v_m)|0\rangle \qquad (3.16)$$

with eigenvalues

$$\Lambda^{(m)} = (\alpha(v))^M \prod_{l=1}^m \frac{1}{f(v_l, v)} + (\delta(v))^M \prod_{l=1}^m \frac{1}{f(v, v_l)} \qquad (3.17)$$

where

$$\alpha(v) = p \exp[\beta E_y(n - 1)/2] \sin[v + (n - 1)\eta] \qquad (3.18a)$$

$$\delta(v) = q \exp[\beta E_y(n - 1)/2] \sin[v - (n - 1)\eta] \qquad (3.18b)$$

and the v_j are determined by the following transcendental equations:

$$\left(\frac{p \sin[v_j + (n - 1)\eta]}{q \sin[v_j - (n - 1)\eta]}\right)^M = \prod_{\substack{l=1 \\ l \neq j}}^m (-2) \frac{\sin(2\eta + v_j - v_l)}{\sin(2\eta + v_l - v_j)}. \qquad (3.19)$$

Note that the vertical field factors out in the eigenvalues given by equation (3.17) and

is absent in equation (3.19). Thus, as expected from the earlier discussion, the vertical field gives a trivial additive contribution to the free energy.

We proceed to derive results for the free energy in the symmetric case. One can now take the thermodynamic limit (·Baxter 1982), and make the usual change of variables. We briefly describe some of the intermediate steps. The distribution of v_j can be replaced by a distribution of κ denoted by $R_n(\kappa)$. This obeys the integral equation

$$R_n(\kappa) = \frac{\sin \tilde{\mu}}{\cosh \kappa - \cos \tilde{\mu}} - \frac{1}{2\pi} \int_{-Q}^{Q} \frac{\sin 2\mu}{\cosh(\kappa - \kappa') - \cos 2\mu} R_n(\kappa') \, d\kappa' \qquad (3.20)$$

where

$$\tilde{\mu} = 2(n-1)\eta + \pi \qquad \kappa = -2iv - i\pi \qquad \text{and} \qquad \mu = 2\eta + \pi. \qquad (3.21)$$

Assuming that the maximum value of $\Lambda^{(m)}$ occurs for $m = \frac{1}{4}nM$, i.e., there are as many down arrows as up, we get $Q = \infty$. In the case $Q = \infty$, (3.20) can, as usual, be solved by Fourier transforms yielding

$$\bar{R}_n(x) = \frac{\sinh(\pi - \tilde{\mu})x}{2 \sinh(\pi - \mu)x \cosh \mu x}. \qquad (3.22)$$

For $n = 2$ this reduces to the six-vertex expression

$$\bar{R}_2(x) = \frac{1}{2} \text{sech } \mu x. \qquad (3.23)$$

The free energy per site $f_n(\eta, v)$ is given by (for the case where the first term in (3.17) is the larger)

$$f_n(\eta, v) = -k_B T \ln \omega_1 + k_B T \int_{-\infty}^{\infty} \frac{\sinh(2\eta - 2v)x \sinh 2(n-1)\eta x}{2\pi \sinh \pi x \cosh(2\eta + \pi)x} \, dx. \qquad (3.24)$$

Using the identity

$$\sinh(2\eta - 2v)x \sinh 2(n-1)\eta x = \sinh 2\eta x \sum_{k=1}^{n-1} \sinh[2\eta - 2(v + 2k\eta - n\eta)]x \qquad (3.25)$$

we find

$$f_n(\eta, v) = \sum_{k=1}^{n-1} f_2[\eta, v + (2k-n)\eta] \qquad (3.26)$$

where f_2 is the usual six-vertex model free energy. In (3.26) the constant parts are assumed to have been adjusted by choosing the normalisation factors appropriately. Apart from changes of variables, this provides an explicit derivation of the result Onody and Karowski (1983) gave. Recall that the phase transitions in the six-vertex model occur as one varies $\Delta = \cos 2\eta$ where η is the first argument of f_2. Thus, all the $(n-1)$ terms exhibit a singularity at the same point. Hence, in the zero-field case the nature of the phase transition is identical to that in the six-vertex model. Note that for all positive vertex weights, the condition (3.5) appears to preclude the antiferroelectric phase transition, i.e., at $\Delta = -1$. However, general symmetry relations which might map unphysical regions of parameter space onto physical regions might exist and allow such transitions.

4. Concluding remarks

We have presented an exact solution of a generalisation of the six-vertex model. We point out that the states of this model correspond to different spin representations of SU(2) with the six-vertex model corresponding to the fundamental representation. The operator B plays the role of the lowering operator J_-. This observation might provide an understanding of the fact that the $\mathscr{L}$ matrix (and hence, all the commutation relations) are identical in these models. The free energy in the absence of external fields is a sum of six-vertex free energies. Our solution also allows for an investigation of the free energy in the presence of fields. It is known that the six-vertex model in a field has a rich phase diagram. The analysis in the presence of fields is complicated and is deferred to a future publication. We also note for $n \geq 3$ models the general solution to the compatibility conditions imposed by (1.8) which allow more freedom in the choice of vertex weights has not been explored.

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